6 Back to Basics

Debts

- ∞-category
- DG-category
- derived stacks
- BG, Bun_G
- $f_*: QC(X) \to QC(Y)$ continuous if f schematic, quasi-cont
- D-modules + De Rham Stack

1) Homotopical Algebra & ∞-categories

algebra	main obj	via	"in practice" res
homological	abelian cat A	$ch(A)$	proj. res (inj res)
homotopical	cat $\mathcal{C}$	$s\mathcal{C}$ simplicial obj in $\mathcal{C}$	cofibrant res (Fibrant res)

- $A \hookrightarrow \underset{\text{chains?}}{ch(A)}$ in deg 0
- $\mathcal{C} \hookrightarrow s\mathcal{C}$ a const. object
- If we regard A as $\mathcal{C}$, then compatibility is given by Dold-Kan correspondence
- D-K is equivalence of model cats
- D-K is equivalence of ∞-cats

Introduce a category Δ, <u>simplex category</u>

<u>obj</u>: $[n] = \{0, \dots, n\}$ $\quad n \in \mathbb{Z}_{\geq 0}$

<u>mor</u>: monotonic, non-decreasing maps

$d^i : [n-1] \to [n]$ "misses" i, $\quad 0 \leq i \leq n$

coface map $\{0, \dots, n\} \to \{0, \dots i-1, i+1, \dots n\}$

$s^j : [n] \to [n-1]$ $\quad 0 \leq j \leq n-1$

codegeneracy map $\{0, \dots, n\} \to \{0, \dots i, i, \dots n-1\}$

<u>Defn</u>| A simplicial set is a functor

$$X : \Delta^{op} \to Set$$
$$[n] \to X([n]) = X_n$$
$$d^i \to d_i = X(d^i) : X_n \to X_{n-1}$$
$$s^j \to s_j = X(s^j) : X_{n-1} \to X_n$$

$$\begin{tikzcd}
X_0 \arrow[r, bend left, "s_0"] & X_1 \arrow[l, bend right, "d^0_0"'] \arrow[l, bend left, "d^0_1"] \arrow[r, bend left, "s_0"] \arrow[r, bend left=60, "s_1"'] & X_2 \arrow[l, bend right=60, "d_0"'] \arrow[l, "d_1"'] \arrow[l, bend left=60, "d_2"] & \cdots \\
\{v_0 \dots v_l\} \arrow[u, equal] & \{e_1 \dots e_m\} \arrow[u, equal] & \{F_1 \dots F_n\} \arrow[u, equal] &
\end{tikzcd}$$

<u>claim</u>| This ↑ diagram encodes the data of X_i.

Any morphism ~~of~~ is a composition of d_i, s_j

~~Still there are~~

~~Claim~~ ~~This~~

still, there are relations:

e.g. $d^i d^{i+1} = d^i d^i$

(check) $\{0,1,\dots\} \to \{0,1,\dots i-1, i+2,\dots\}$

Exer (Simplicial identities)
There are more than these. Find them all.

To find the shape of a simplicial set X we define $|\cdot| : \mathrm{Set}_\Delta \to \mathrm{Top}$ "geometric realization"

$$X \to \coprod_{n \in \mathbb{Z}_{\geq 0}} X_n \times |\Delta^n| \Big/ \sim$$

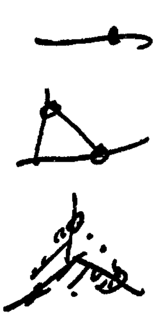
n-simplex

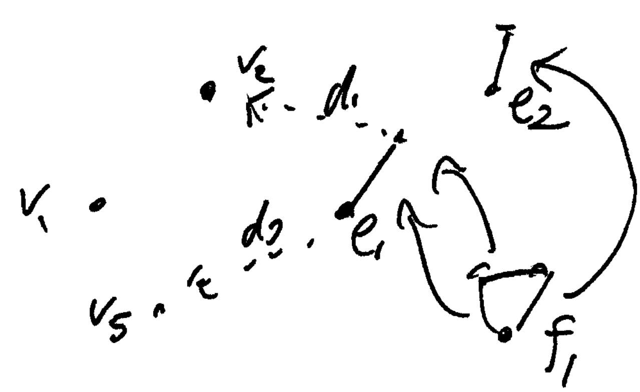
$$|\Delta^n| = \{ t_0 \dots t_n \in \mathbb{R}^{n+1} : t_i \geq 0, \ \sum t_i = 1 \}$$

$n=0$

$n=1$

$n=2$





Question: Is there a simplicial set Δ^n s.t. $|\cdot|(\Delta^n) = |\Delta^n|$

Answer: $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ Yoneda Functor

$X_n = \mathrm{Hom}(\Delta^n, X)$

$Sing : Top \to Set$

$Y \to Sing\, Y_n = Hom_{Top}(|\Delta^n|, Y)$

$$sSet \underset{sing}{\overset{|-|}{\rightleftarrows}} Top$$

<u>convention:</u> $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ F is left-adjoint
G is right-adjoint

$$\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "G"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$$ sometimes

<u>Defn</u> A simplicial object in $\mathcal{C}$ is a functor
$\Delta^{op} \to \mathcal{C}$

Ex: $\mathcal{C} = Set \rightsquigarrow sSet$
$\mathcal{C} = Ab \rightsquigarrow sAb$ simplicial abelian groups
$\mathcal{C} = Ring \rightsquigarrow sRing$ simplicial Rings

<u>Dold Kan</u>

Fix A abelian cat. (e.g. $A = Ab$)
consider $sA \ni A$ A_n is abelian gp $\forall n \in \mathbb{Z}_{n \geq 0}$

<u>Defn</u> A Moore (unnormalized) chain complex associated to A is $C(A)$ where

$$\begin{cases} C(A)_n = A_n \\ d|_{A_n} = d_0 - d_1 + \cdots + (-1)^n d_n \end{cases}$$

<u>Exer:</u> $d^2 = 0$

<u>Defn</u> A normalized chain complex $N(A)$ is defined as:

$$\begin{cases} N(A)_n = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1}) \\ d = d_0 \end{cases}$$

$N(A) \subset C(A)$ subcomplex

$ch_{\geq 0} A = \{$ chain complexes in A concentrated in non-negative degrees

$(\cdots \to C_2 \to C_1 \to C_0 \to 0)$

<u>Thm</u> Dold-Kan

$\exists$ equivalence $\quad sA \underset{\sim}{\xrightarrow{N}} ch_{\geq}(A)$

s.t. $\pi_n(A) \cong H_n(NA)$

"Idea for homotopy $\simeq$ homology"

$\Delta^n / \partial\Delta^n \to A \qquad \mathrm{Hom}_{sSet}(\Delta^n, A) = A_n$

$\sim S^n \to A \quad \cdots\to \quad \mathrm{Hom}_{sSet}(\Delta^n/\partial\Delta^n, A)$

$= \bigcap_{i=0}^{n} \ker(d_i : A_n \to A_{n-1})$

$$\begin{tikzcd} \Delta^n/\partial\Delta^n \arrow[r] \arrow[d, "d^0"'] & A \\ \Delta^{n+1}/\Lambda_0^{n+1} \arrow[ur] & \end{tikzcd}$$

$\longleftrightarrow$ null-homotopic maps

$n=1 \qquad |: \to S^1$

Model Categories

Idea / Given a category $\mathcal{C}$,
sometimes one might want to regard
some morphisms as if they were iso,
Model categories are supposed to help us!

Top $\ni X, Y$
$f: X \to Y$ is called a weak homotopy equivalence
if $\pi_n f: \pi_n X \to \pi_n Y$ agrees $\forall n$

Ho(Top) obj: top'l spaces
mor: cont. maps
Forcing weak homotopy equivalence
as homotopy equivalence

There is a nice class of spaces:

Thm / (Whitehead)
If X, Y are CW then $f: X \to Y$ weak homotopy e[quivalence]
is a homotopy equi[valence]

Thm / (CW approximation)
For $X \in$ Top, $\exists$ CW complex QX
s.t. $QX \xrightarrow{\sim} X$ by weak homotopy

Ho(Top)

$\parallel$

CW complexes, homotopy equivalence.

$\mathrm{Hom}_{Ho(Top)}(X, Y)$

$\wr\downarrow$

$\mathrm{Hom}_{Top}(QX, QY)$

Take $\mathcal{C}$, with W the weak equivalences

$\mathcal{C}[W^{-1}] = Ho(\mathcal{C})$

In model category theory, we find classes of morphisms called
- W - weak equivalences
- fibrations
- cofibrations

They satisfy axioms:

category	Nice class of spaces	Approximation
Top	CW complexes	CW approx
CW	cofibrant-fibrant objects	$QX \xrightarrow{\sim} X$
sSet	Kan complexes	$QX \xrightarrow{\sim} X$
$ch_{\geq 0}$ $ch_{\leq 0}$	inj. modules proj. modules	$QX \xrightarrow{\sim} X$

like co-ordinate charts

2) DG categories

$DGCat_{cont.}$ presentable stable (cocomplete) having all limits

$\left\{ \begin{array}{l} \text{dg categories} \\ \text{Functors are continuous, preserving colimits} \end{array} \right.$

$\mathcal{C}_1, \mathcal{C}_2 \in$ DG Categories

want $\mathcal{C}_1 \otimes \mathcal{C}_2$

is a DG category enriched over Vect

Cat	Vect	$DGCat_{cont}$
	linear	cocomplete
	$V_1 \times V_2 \to W$ bilinear $\Leftrightarrow V_1 \otimes V_2 \to W$	$\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ bi-continuity $\mathcal{C}_1 \otimes \mathcal{C}_2 \xrightarrow{cont.} \mathcal{D}$
Unit	k	vect

vect ⇑ vector spaces with chain complex structure!!

producti

$$\begin{tikzcd} & W \arrow[d, dotted] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & C_1 \otimes C_2 \arrow[dl] \arrow[dr] & \\ C_1 & & C_2 \end{tikzcd}$$

Defn | $\mathcal{C}$ complete$_*$ DG category

- An object $c \in \mathcal{C}$ is called compact if $\mathrm{Hom}_{\mathcal{C}}(c,-) : \mathcal{C} \to \mathrm{Vect}$ is continuous
- $\mathcal{C}$ is called compactly ~~continuous~~ generated if

$\exists$ compact objects c_α generating $\mathcal{C}$

$\underline{\mathrm{Hom}}_{\mathcal{C}}(c_\alpha, C) = 0 \quad \forall \alpha \in I$

internal hom $\Rightarrow C = 0$

Ex | $\mathrm{Vect}^\heartsuit$ the abelian cat of vector spaces

For which V is $\mathrm{Hom}_{\mathrm{Vect}^\heartsuit}(V,-)$ continuous?

$$V = \operatorname{colim}_{i \in I} V_i$$

$$\underline{\mathrm{Hom}}(\operatorname{colim} X_i, Y) = \lim \underline{\mathrm{Hom}}(X_i, Y)$$

$$\underline{\mathrm{Hom}}(\lim X_i, Y) \longleftarrow \operatorname{colim} \underline{\mathrm{Hom}}(X_i, Y)$$

$$\underline{\mathrm{Hom}}(X, \operatorname{colim} Y_i) \longleftarrow \operatorname{colim} \underline{\mathrm{Hom}}(X, Y_i)$$

$$\underline{\mathrm{Hom}}(X, \lim Y_i) = \lim \underline{\mathrm{Hom}}(X, Y_i)$$

V is ~~[illegible]~~ compact $\Leftrightarrow$ V is f.d.

$\Rightarrow$ $\operatorname{colim} \underline{\mathrm{Hom}}(V_i, V_i) \xrightarrow{\sim} \underline{\mathrm{Hom}}(V, V)$

$f_i \; V \to V_i \xrightarrow{\sim} id$

$i : V_i \to V$

$i \circ f = id_V \Rightarrow \dim V < \infty$ } think about this!

$\Leftrightarrow$ $\mathrm{Hom}(V, -) = V^* \otimes$ left adj. $\blacksquare$

$\mathcal{C}$ cocomplete DG

$\mathcal{C}^c$ full subcat of compact objects

<u>claim</u> $\mathcal{C}^c$ knows almost everything about $\mathcal{C}$

$\mathcal{C}^0$ small category $\rightsquigarrow$ $\mathrm{Ind}(\mathcal{C}^0)$ ind-completion

s.t. $\mathrm{Funct}(\mathcal{C}^0, \mathcal{D}) \simeq \mathrm{Funct}_{cont.}(\mathrm{Ind}(\mathcal{C}^0), \mathcal{D})$

<u>Thm</u> $\mathcal{C}^c \to \mathcal{C}$ full

$\mathrm{Ind}(\mathcal{C}^c) \xrightarrow{\sim} \mathcal{C}$ is an equivalence

$\Leftrightarrow$ $\mathcal{C}$ is compactly generated

$S = \mathrm{Spec}\, A$ affine derived scheme

A is a derived ring

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scheme : Ring $\to$ Set

Prestack : Derived Ring $\to$ Derived Set

$sAb \simeq ch_{\geq 0} \simeq ch^{\leq 0}$

derived $\rightsquigarrow$ simplicial

$sRing \simeq cdga^{\leq 0}$

if $\mathcal{C} = QC(S) = A\text{-mod}$

<u>Claim</u> $\mathcal{F} \in QC(S)$ is compact

$\Leftrightarrow$ $\mathcal{F}$ is perfect
$\mathcal{F} \in \mathrm{Perf}(S)$ } what does this mean?

<u>Defn</u> $S = \mathrm{Spec}\, A$ is almost of finite type if A is
is $H^0(A)$ is of finite type $/k$
$H^i(A)$ is of finite type $/ H^0(A)$

If S-classical
$A \in \mathrm{Ring}$

A <u>perfect complex</u> is a finite complex of vector bundles on S.

In general perfect complex includes $\mathcal{O}_S$ and is closed under finite limits, colimits, sums.

$A = k[\epsilon]/(\epsilon^2)$
k is A-module

$[\cdots \to A \xrightarrow{\cdot\epsilon} A \xrightarrow{\cdot\epsilon} A] \to k$

k is not perf complex
$k \in QC(S)$
but $k \notin \mathrm{Perf}(S)$

<u>Defn</u> $\mathcal{F} \in \mathrm{Coh}(S) \subset QC(S)$

$\Leftrightarrow$ $\mathcal{F}$ is cohomologically bounded and each cohomology is coherent over $H^0(A)$
(finitely presented)

Ex (again) $A = k[\epsilon]/(\epsilon^2)$

$k \in Coh(S) \subset QC(S)$

$k \notin Perf(S)$

$Perf(S) \not\subset Coh(S)$

$\not\ni \mathcal{O}_S$

$= S \in cdga^{\leq 0}$

$A = k[u] \quad \deg u = -2$

Defn| $S = Spec\ A$ is <u>eventually</u> <u>coconnective</u> if $H^i(A) = 0$ for $i \ll 0$

IF S is eventually coconnective then $Perf(S) \hookrightarrow Coh(S)$

Defn| S is of affine type if it's of almost finite type & eventually coconnective

Defn| Let S be of affine type

$I.C. := Ind(Coh(S)) \qquad Coh(S) \to QC(S)$

$$QC(S) \underset{\Psi_S}{\overset{\Xi_S}{\rightleftarrows}} IC(S)$$

$QC(S) = Ind(Perf(S))$

Lemma|

- F is continuous
- IF G is continuous, F sends cpt. to cpt.

If $\mathcal{C}$ compactly generated, the converse holds

Why I.C., not QC?

$f : X \to Y$

$f_* : QC(X) \to QC(Y)$

If f is proper then one expects its right adjoint is $\underset{\sim}{f^!}$

$\mathrm{Perf} \xrightarrow{f_*} \mathrm{Perf}$ <u>False</u>

$\mathrm{Coh} \xrightarrow{f_*} \mathrm{Coh}$ true

$\rightsquigarrow f^!$ is the natural functor to consider.